

Spin Path Integral And Quantum Mechanics In Rotating Reference Of Frame

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In the present paper, we develop a path integral formalism of the quantum mechanics in a rotating reference of frame, and apply this description to Rabi oscillation, which is important for realizing the quantum qubit, and to the explanations of various experiments related to Sagnac effect. In particular, we propose a path integral description to spin degrees of freedom, connect it to the Schwinger bosons realization of angular momentum, and then apply this description to path integral quantization of spin in the rotating frame.

1. Introduction

The Quantum mechanics in a rotating reference of frame has many important applications, for examples, to Rabi oscillation which is crucial for cavity quantum electro-dynamics [1] and for designing the qubit circuit of a scalable quantum computer (e.g. [2]), to the unified geometric phase and spin-Hall effect in optics by analyzing the Coriolis effect [3].

The main purpose of the present paper is to develop a path integral description of the quantum mechanics in rotating frame. We investigate a charged particle in the rotating frame with a uniform external magnetic field applied to it, and use the path integral description to explain various related experiments, e.g. the sagnac effect [5] due to the coupling between the orbital angular momentum of the particle and the rotation of the reference frame (see [6] for some important applications of this effect), and the spin-rotation coupling analog of this effect, in Neutron interference [7] (see [8] for the experimental observation of the phase shift via spin-rotation coupling), to solve Rabi oscillation problem which is traditionally solved in Hamiltonian formulation [9].

To achieve these purposes, we present the quantum mechanical theory of spin degrees of freedom in rotating frame by using the spin path integral description (see e.g. [10]) and connecting it to the Schwinger bosons realization of the algebra of angular momentums \vec{J} . Combining with path integral description to coordinate variables, we can develop the full path integral theory of a point particle in a rotating frame. We can then derive the phase factor due to the spin-rotation coupling and its orbital angular momentum counterpart (the sagnac effect), give a path integral solution for Rabi oscillation problem, and extend the Coriolis force from classical mechanics to quantum mechanics, by using the quantum action principle [11].

2. The Path Integral Formalism of Quantum Mechanics In Rotating Frame

Spin Path Integral And Schwinger Bosons

To describe the spin degrees of freedom of a non-relativistic spin $s = \frac{n}{2}$ particle with mass m and electric charge e , we introduce the action

$$I = \int dt \left[i\phi^\dagger \frac{d}{dt} \phi - \lambda(\phi^\dagger \phi - n) \right], \quad (2.1)$$

where ϕ is a two components bosonic variable, $\phi^\dagger = (\overline{\phi^1}, \overline{\phi^2})$ is its Hermitian conjugation, and λ is a Lagrangian multiplier. (2.1) has a $U(2)$ symmetry which acts as $\phi \rightarrow U\phi$, where

$U \in U(2)$. To realize the $SU(2)$ symmetry of spin, one should modulo out a $U(1)$ factor, which acts as $\phi \rightarrow e^{-i\theta}\phi, \bar{\phi} \rightarrow e^{i\theta}\bar{\phi}$, by requiring all the physical observable being $U(1)$ invariant. The conservation charge of this $U(1)$ transformation is $N(\phi) = \phi^\dagger\phi$ which is itself a physical observable. Other fundamental physical observable¹ are the conservation charges $\vec{S}(\phi) = \frac{1}{2}\phi^\dagger\vec{\sigma}\phi$ of the $SU(2)$ symmetry, where $\vec{\sigma}$ are the three Pauli matrixes. In the quantum theory $\vec{S}(\phi)$ realize the $SU(2)$ symmetry algebra, which we identify with the spin, thus ϕ is a Pauli spinor. In the path integral quantization, $\vec{S}(\phi)$ can be inserted in the path integral $\int \mathcal{D}\phi\mathcal{D}\bar{\phi}\mathcal{D}\lambda \exp(iI/\hbar)$.

To see the connection between above spin path integral and the $SU(2)$ algebra more clearly, we now perform the canonical quantization procedure. Clearly, $i\phi^\dagger$ is the canonical momentum of ϕ , and the Hamiltonian for the free spin is trivial. ϕ is now realized as the Schwinger bosons with the commutators $[\phi^\alpha, \bar{\phi}^\beta] = \hbar\delta_{\alpha\beta}$, where $\alpha, \beta = 1, 2$ are the indexes of ϕ . The constraint equation associated with λ will restrict us to the Hilbert space of n Schwinger bosons $(\phi^\dagger\phi - n)|\psi\rangle = 0$, where $|\psi\rangle$ is an arbitrary state of the Hilbert space \mathcal{H}_s of the spin, $|\psi\rangle \in \mathcal{H}_s$. The basis of \mathcal{H}_s can be constructed by acting n creation bosons $\bar{\phi}^\alpha$ on the Fock vacuum², $|\alpha_1\alpha_2...\alpha_n\rangle = \bar{\phi}^{\alpha_1}\bar{\phi}^{\alpha_2}...\bar{\phi}^{\alpha_n}|0\rangle$, where $|0\rangle$ is the Fock vacuum which satisfy $\phi^\alpha|0\rangle = 0$. The spin operators \vec{S} are now realized as

$$\vec{S} = \frac{1}{2}\phi^\dagger\vec{\sigma}\phi. \quad (2.2)$$

The action of \vec{S} on \mathcal{H}_s gives out $s = \frac{n}{2}$ representation.

Spin In The Rotating Reference of Frame

We now turn to the non-inertial reference of frame \mathcal{S} , which is rotating with angular velocity $\vec{\omega}(t)$. Firstly, we note that in terms of the variables in \mathcal{S} , the time changing of spinor $\phi(t)$ should be

$$\frac{d\phi}{dt} - i\frac{1}{2}\vec{\omega}(t) \cdot \vec{\sigma}\phi, \quad (2.3)$$

which is the generalization of the ordinary $\frac{d\vec{v}}{dt} + \vec{\omega}(t) \times \vec{v}$ for the time changing—in terms of variables of \mathcal{S} —of vector $\vec{v}(t)$. This generalization is motivated by noticing that under an

¹ A little thought show us that, an arbitrary physical observable can be written as the function of $\vec{S}(\phi)$ and $N(\phi)$.

² In the full Fock space, one can also construct the spin coherent state $|z\rangle$ by defining it as the eigenstates of ϕ , with $\phi^\alpha|z\rangle = z^\alpha|z\rangle$. The mean value of the spin operator \vec{S} on the spin coherent states is $\langle z|\vec{S}|z\rangle = \frac{1}{2}z^\dagger\vec{\sigma}z$.

infinitesimal rotation $(1 - i\delta\vec{\theta} \cdot \vec{J})$, a vector \vec{v} transforms as $\vec{v} \rightarrow \vec{v} - i(\delta\vec{\theta} \cdot \vec{L})\vec{v} = \vec{v} + \delta\vec{\theta} \times \vec{v}$, and a spinor ϕ transforms as $\phi \rightarrow \phi - i\frac{1}{2}\delta\vec{\theta} \cdot \vec{\sigma}\phi$.

Thus, in frame \mathcal{S} , the action for the spin is given by replacing the $\frac{d\phi}{dt}$ of (2.1) with $\frac{d\phi}{dt} - i\frac{1}{2}\vec{\omega}(t) \cdot \vec{\sigma}\phi$

$$\begin{aligned} I_S &= \int dt \left[i\phi^\dagger \left(\frac{d}{dt} - i\frac{1}{2}\vec{\omega}(t) \cdot \vec{\sigma} \right) \phi - \lambda(\phi^\dagger \phi - n) \right] \\ &= \int dt \left[i\phi^\dagger \frac{d}{dt} \phi + \vec{\omega}(t) \cdot \frac{1}{2}\phi^\dagger \vec{\sigma} \phi - \lambda(\phi^\dagger \phi - n) \right]. \end{aligned} \quad (2.4)$$

In terms of the path integral quantization, spin in the rotating frame is described as an insertion $\vec{S}(\phi) = \frac{1}{2}\phi^\dagger \vec{\sigma} \phi$ in $\int \mathcal{D}\phi \mathcal{D}\bar{\phi} \mathcal{D}\lambda \exp(iI_S/\hbar)$.

To compare with the Hamiltonian formalism, one canonically quantize (2.4). The Hamiltonian H_S can be easily got³,

$$H_S = -\vec{\omega}(t) \cdot \vec{S}. \quad (2.5)$$

This result is exactly identical with [9][5].

In the static frame, one should insert $\exp(ig\frac{\mu}{\hbar} \int dt \vec{B}(t) \cdot \vec{S}(\phi))$ into the path integration to account for the spin-magnetism coupling, where $\mu = e/2mc$, and g is the g factor of the considered particle, and $\vec{B}(t)$ is a uniform magnetic field. The total effect is to change the action from I for the free spin to I_B for the spin in \vec{B} field, where I_B is given as

$$I_B = \int dt \left[i\phi^\dagger \frac{d}{dt} \phi - \lambda(\phi^\dagger \phi - n) + g\mu \vec{B}(t) \cdot \vec{S}(\phi) \right]. \quad (2.6)$$

We now generalize the consideration to the rotating frame. For simplicity, we assume that the angular velocity $\vec{\omega}$ is time independent, and use $\vec{B}(t)$ to denote the magnetic field in the rotating frame \mathcal{S} . To write down the explicit form of the magnetic field in the static inertial frame, we first decompose $\vec{B}(t)$ into the parallel part $\vec{B}_\parallel(t)$ —which is parallel to $\vec{\omega}$ —and the perpendicular part $\vec{B}_\perp(t)$ —which is perpendicular to $\vec{\omega}$. The magnetic field in the static frame can then be written as $\vec{B}_{iner}(t) = \vec{B}_\parallel(t) + \vec{B}_\perp^+(t)e^{i\omega t} + \vec{B}_\perp^-(t)e^{-i\omega t}$, where the additional factors $e^{i\omega t}$ and $e^{-i\omega t}$ are due to the rotation of \mathcal{S} .⁴ Now, the action (2.4) and action (2.6) should be combined into the action I_{BS} ,

$$I_{BS} = \int dt \left[i\phi^\dagger \frac{d}{dt} \phi + (\vec{\omega} + g\mu \vec{B}(t)) \cdot \vec{S}(\phi) - \lambda(\phi^\dagger \phi - n) \right]. \quad (2.7)$$

³ The quantization relation of the Schwinger bosons, the constraint equation and the spin Hilbert space \mathcal{H}_s are all the same as mentioned in above subsection.

⁴ To write out the expressions of \vec{B}_\perp^+ and \vec{B}_\perp^- explicitly, one can coordinates the plane perpendicular to $\vec{\omega}$ as $x - y$ plane, then $\vec{B}_\perp^+ = \vec{B}_\perp^x + i\vec{B}_\perp^y$ and $\vec{B}_\perp^- = \vec{B}_\perp^x - i\vec{B}_\perp^y$.

In terms of the canonical quantization, the Hamiltonian in \mathcal{S} is H_{BS} , with

$$H_{BS} = -g\mu\vec{B}(t) \cdot \vec{S} - \vec{\omega} \cdot \vec{S}, \quad (2.8)$$

in agreement with [9].

Include Position Variables

We now include the position variables \vec{x} of the particle in the rotating frame \mathcal{S} . The spatial part I_x of the total action is given as

$$\begin{aligned} I_x &= \int dt \left[\frac{1}{2}m \left(\frac{d\vec{x}}{dt} + \vec{\omega} \times \vec{x} \right)^2 - V(\vec{x}) + e \left(\frac{1}{2}\vec{B} \times \vec{x} \right) \cdot \left(\frac{d\vec{x}}{dt} + \vec{\omega} \times \vec{x} \right) / c \right] \\ &= \int dt \left[\frac{1}{2}m \left(\frac{d\vec{x}}{dt} \right)^2 + m(\vec{\omega} + \mu\vec{B}) \cdot (\vec{x} \times \frac{d\vec{x}}{dt}) \right] \\ &\quad - \int dt \left[V(\vec{x}) - \frac{1}{2}m(\vec{\omega} \times \vec{x})^2 - m\mu(\vec{B} \times \vec{x}) \cdot (\vec{\omega} \times \vec{x}) \right], \end{aligned} \quad (2.9)$$

where $V(\vec{x})$ stands for the potential energy of the particle in static inertial frame, $-\frac{1}{2}\vec{B} \times \vec{x}$ is the vector potential $\vec{A}(\vec{x})$. In the path integral quantization, one should include the factor $\int \mathcal{D}\mathbf{x} \exp(iI_x/\hbar)$.

We can quantize (2.9) canonically. Clearly, the canonical momentums of \vec{x} are $\vec{p} = m \left(\frac{d\vec{x}}{dt} + \vec{\omega} \times \vec{x} + \mu\vec{B} \times \vec{x} \right)$. The Hamiltonian can be written out

$$H_x = \frac{\vec{p}^2}{2m} - (\mu\vec{B} + \vec{\omega}) \cdot \vec{L} + V_{eff}(\vec{x}), \quad (2.10)$$

where $\vec{L} = \vec{x} \times \vec{p}$ is the angular momentum operator, and the effective potential $V_{eff}(\vec{x})$ is given by $V_{eff}(\vec{x}) = V(\vec{x}) + \frac{1}{2}\mu^2m(\vec{B} \times \vec{x}) \cdot (\vec{B} \times \vec{x})$.

Spin-Orbital Coupling

We'll now discuss the coupling between the spin and the spacial variables by treating it as a perturbation to action $I_{BS} + I_x$. In the static frame, the simplest such term that preserves the parity is $\xi(\vec{x})\vec{S}(\phi) \cdot [\vec{x} \times (\frac{d\vec{x}}{dt} + e\frac{1}{2}\vec{B}_{iner} \times \vec{x})]$, where the small factor $\xi(\vec{x})$ is a somewhat arbitrary function of \vec{x} , for the case of H atom, $\xi(\vec{x}) = \frac{1}{2mc^2} \frac{1}{|\vec{x}|} \frac{dV_C(\vec{x})}{d|\vec{x}|}$. Now, go to the \mathcal{S} frame, we'll have the term $I_{LS} = \int dt \xi(\vec{x}) [\vec{x} \times (\frac{d\vec{x}}{dt} + \vec{\omega} \times \vec{x} + e\frac{1}{2}\vec{B} \times \vec{x})] \cdot \vec{S}(\phi)$. In terms of path integral quantization, one should insert $\exp(iI_{LS}/\hbar)$ in $\int \mathcal{D}\phi \mathcal{D}\bar{\phi} \mathcal{D}\lambda \mathcal{D}\mathbf{x} \exp\{i\frac{1}{\hbar}(I_{SB} + I_x)\}$. In terms of canonical quantization, the spin-orbital coupling is just the usual perturbation $V_{LS} = \xi(\vec{x})\vec{L} \cdot \vec{S}$ to the potential energy of the system.

3. Neutron Interference, And Rabbi Oscillating

We now apply the general formalism in above section to give unified interpretations of some well known quantum mechanical effects or experiments concerning rotating reference of frame.

Neutron interference

In the neutron interference experiment Earth is the rotating frame \mathcal{S} . The coupling between the angular momentum and rotation of the frame will course a phase shift $\Delta\varphi$. The relevant terms of the action is the kinematic terms $I_k = \int dt \left[\frac{1}{2}m \left(\frac{d\vec{x}}{dt} \right)^2 + i\phi^\dagger \frac{d}{dt}\phi - \lambda(\phi^\dagger\phi - n) \right]$ plus the terms account for the inertial force $I_r = \int dt \left[m(\vec{x} \times \frac{d\vec{x}}{dt}) + \vec{S}(\phi) \right] \cdot \vec{\omega}$. In the interference experiment, the insertion of $\exp(iI_r/\hbar)$ in the path integration will contribute a phase factor $\exp(i\frac{\vec{S}}{\hbar} \cdot \oint dt \vec{\omega}) \exp(i\frac{2m\vec{\omega}}{\hbar} \cdot \frac{1}{2} \oint_C (\vec{x} \times d\vec{x}))$, where $\frac{1}{2} \oint_C (\vec{x} \times d\vec{x})$ is just the area \vec{A}_C surrounded by the path of the neutron, and $\oint dt \vec{\omega}$ equals to $2T\vec{\omega}$, where T is the flying time of the neutron. Thus, the total phase shift $\Delta\varphi = \frac{1}{\hbar} 2(m\vec{A}_C + T\vec{S}) \cdot \vec{\omega}$, where the first term is the so called Sagnac effect. In terms of the Hamiltonian formalism, the relevant terms are $H_r = -\vec{\omega} \cdot (\vec{L} + \vec{S})$, which has been derived to explain the interference experiment.

Rabbi Oscillating

In the nuclear-magnetism resonance experiment. A magnetic momentum $g\mu_B$ is coupled to a control magnetic field \vec{B}_\parallel and a transverse rotating field $\vec{B}_\perp^+ e^{i\omega t} + \vec{B}_\perp^- e^{-i\omega t}$. In the frame \mathcal{S} rotating along the direction of \vec{B}_\parallel with angular velocity ω , this system is easy to solve. The relevant terms of the action are the kinematic terms $I_{Sk} = \int dt \left[i\phi^\dagger \frac{d}{dt}\phi - \lambda(\phi^\dagger\phi - n) \right]$, plus the terms $I_{Br} = \int dt (\vec{\omega} + g\mu\vec{B}) \cdot \vec{S}(\phi)$. At the resonant frequency $\vec{\omega} + g\mu\vec{B}_\parallel = 0$, the factor $\exp\{\frac{i}{\hbar} \int dt (g\mu\vec{B}_\perp \cdot \vec{S})\}$ will cause the state of the spin oscillating between the up and down state with oscillating frequency $\omega_R = \frac{g\mu|\vec{B}_\perp|}{2\hbar}$. In the more general case the oscillating frequency Ω is given by $\Omega^2 = \left(\frac{\vec{\omega} + g\mu\vec{B}_\parallel}{2\hbar} \right)^2 + \omega_R^2$. All these results can also be derived from the Hamiltonian $H_{BS} = -g\mu\vec{B} \cdot \vec{S} - \vec{\omega} \cdot \vec{S}$.

Equation Of Motions

We now digress to discuss the equation of motions—in the rotating frame \mathcal{S} —derived by using Schwinger's quantum action principle to see the quantum extension of the ordinary non-inertial force in the classical theory, e.g. the Coriolis force.

The relevant action is $I = \int dt \left[\frac{1}{2}m \left(\frac{d\vec{x}}{dt} \right)^2 + m\vec{\omega} \cdot (\vec{x} \times \frac{d\vec{x}}{dt}) - V(\vec{x}) + \frac{1}{2}m(\vec{\omega} \times \vec{x})^2 \right]$. The quantum action principle tells us $\langle \psi_f | \delta I | \psi_i \rangle = 0$. Carrying out the variation of I will give us the equation of motion $\langle \psi_f | m \frac{d^2 \vec{x}}{dt^2} + 2m \frac{d\vec{x}}{dt} \times \vec{\omega} + m(\vec{\omega} \times \vec{x}) \times \vec{\omega} | \psi_i \rangle = -\langle \psi_f | \frac{\partial V(\vec{x})}{\partial \vec{x}} | \psi_i \rangle$, which is just the quantum mechanical generalization of the ordinary Newton equation in rotating frame, where the Coriolis force $2m\vec{\omega} \times \frac{d\vec{x}}{dt}$ comes from the $\int dt m\vec{\omega} \cdot (\vec{x} \times \frac{d\vec{x}}{dt})$ term of the action I . In terms of the Hamiltonian formalism. The Coriolis force comes from the $-\vec{\omega} \cdot \vec{L}$ term of the Hamiltonian $H = \frac{\vec{p}^2}{2m} - \vec{\omega} \cdot \vec{L} + V(\vec{x})$.

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